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On the Localization Property of Multiple Fourier Series*

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1. Let T_n be the unit cube in the n-dimensional Euclidean space E_n, that is,

$$T_n = \{x = (x_1, \ldots, x_n) \colon -\pi \leqslant x_j < \pi, \qquad j = 1, \ldots, n\}.$$

Denote by $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ points of T_n, and by $m = (m_1, \ldots, m_n)$ lattice point of E_n. For a function f of $L^1(T_n)$, its Fourier series is defined by

$$S(x, f) = \sum_m \hat{f}_m e^{im\cdot x}$$

where

$$\hat{f}_m = \frac{1}{(2\pi)^n} \int_{T_n} f(x)\, e^{-im\cdot x}\, dx,$$

$$m\cdot x = m_1 x_1 + \ldots + m_n x_n, \qquad dx = dx_1 \ldots dx_n$$

and $\sum$ runs over all lattice points.

Let $W_R(x,f)$ be a summation method of the Fourier series of f. Then we say that W_R has the localization property (for abbreviation, L.P.) for $L^p(T_n)$, if for any $f \in L^p(T_n)$ vanishing on an open set, $W_R(x,f)$ converges uniformly to zero on each compact set contained in the open set.

A classical theorem of Riemann states that in the one-dimensional case, partial sums of Fourier series have the localization property for $L^1(-\pi, \pi)$. We shall investigate the n-dimensional analogues of this theorem. In the following we assume always $n \geqslant 2$.

2. Let $l = (l_1, \ldots, l_n)$ be a lattice point with non-negative coordinates, and let the lth partial sum of the Fourier series of f be

$$s_l(x, f) = \sum \hat{f}_m e^{im\cdot x}$$

where $m = (m_1, \ldots, m_n)$ runs over all m such that $|m_j| \leqslant l_j$, $j = 1, \ldots, n$. Then we have

$$s_l(x, f) = \frac{1}{\pi^n} \int_{T_n} f(y)\, D_l(x - y)\, dy,$$

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where $$D_l(x) = D_{l_1}(x_1) \dots D_{l_n}(x_n)$$

and $$D_{l_j}(x_j) = \sin(l_j + \tfrac{1}{2})\, x_j / 2 \sin(x_j/2).$$

The (C, 1)-mean is defined similarly by

$$\sigma_l(x,f) = \frac{1}{(l_1+1)\dots(l_n+1)} \sum s_m(x,f)$$

$$= \frac{1}{\pi^n} \int_{T_n} f(y)\, K_l(x-y)\, dy,$$

where $$K_l(x) = K_{l_1}(x_1) \dots K_{l_n}(x_n)$$

and $$K_{l_j}(x_j) = 2\{\sin \tfrac{1}{2}(l_j+1)\, x_j / 2 \sin(x_j/2)\}^2/(l_j+1).$$

We call $s_l(f)$ a square partial sum if $l_1 = \dots = l_n$, and a rectangular partial sum for arbitrary l_j.

THEOREM 1. (1) *Square partial sums do not have L.P. for $C(T_n)$.* (2) *Square* (C, 1)*-sums have* L.P. *for $L^p(T_n)$ if $p \geqslant n-1$, but not if $n-1 > p \geqslant 1$.*

Remark 1. Rectangular (C, 1)-sums have L.P. for $C(T_n)$ but not for $L^p(T_n)$, $p > 1$ (see [3], p. 304).

Proof. (1) We show that there exists a function f of $C(T_n)$ which vanishes on a neighborhood of the origin and satisfies

$$\limsup_{j\to\infty} |s_{(j,\dots,j)}(0,f)| = \infty.$$

Let $0 < \epsilon < \delta < \pi$ and let ϕ be a function of $C(T_n)$ such that $\phi(x) = 0$ for $|x| < \epsilon$ and $\phi(x) = 1$ for $x \in T_n$, $|x| > \delta$. Let us put $U_j(f) = s_{(j,\dots,j)}(0, \phi f)$. If our assertion did not hold, then $U_j(f)$ would be bounded for each f of $C(T_n)$. Since the U_j are bounded linear functionals on $C(T_n)$ and their norms are

$$\frac{1}{\pi^n} \int_{T_n} |\phi(y)\, D_{(j,\dots,j)}(y)|\, dy,$$

by the uniform boundedness theorem we get

$$\limsup_{j\to\infty} \frac{1}{\pi^n} \int_{T_n} |\phi(y)\, D_{(j,\dots,j)}(y)|\, dy < \infty.$$

On the other hand, the above integrals are minorized by

$$\frac{1}{\pi^n}\int_\delta^\pi \left|\frac{\sin(j+1/2)y_1}{\sin(y_1/2)}\right| dy_1 \int_0^\pi \left|\frac{\sin(j+1/2)y_2}{\sin(y_2/2)}\right| dy_2 \cdots \int_0^\pi \left|\frac{\sin(j+1/2)y_n}{\sin(y_n/2)}\right| dy_n$$

$$\geqslant A \int_{(j+1/2)\delta}^{(j+1/2)\pi} \frac{|\sin y_1|}{y_1}\, dy_1 (\log j)^{n-1}$$

$$\geqslant A'(\log j)^{n-1}$$

for some constants A, A'. The last term is unbounded as $j \to \infty$ if $n \geqslant 2$, which is absurd.

To prove the second part of (2), it is sufficient to show that

$$I_j = \frac{1}{\pi^n}\int_{T_n} |\phi(y) K_{(j,\ldots,j)}(y)|^q \, dy$$

is unbounded as $j \to \infty$, where $1/p + 1/q = 1$. We may assume $q < \infty$. By the definitions of ϕ and $K_{(j,\ldots,j)}$, we have

$$I_j \geqslant \frac{1}{\pi^n (j+1)^{qn}} \int_\delta^\pi \left\{\frac{\sin(j+1)\dfrac{y_1}{2}}{\sin\dfrac{y_1}{2}}\right\}^{2q} dy_1 \int_0^\pi \left\{\frac{\sin(j+1)\dfrac{y_2}{2}}{\sin\dfrac{y_2}{2}}\right\}^{2q} dy_2$$

$$\cdots \int_0^\pi \left\{\frac{\sin(j+1)\dfrac{y_n}{2}}{\sin\dfrac{y_n}{2}}\right\}^{2q} dy_n.$$

The first integral is larger than some constant multiple of

$$\int_\delta^\pi \frac{\sin^{2q}\left(\dfrac{j+1}{2}\right) y_1}{y_1^{2q}}\, dy_1 \geqslant A j^{2q-1} \int_{j\delta}^{2\pi j} \frac{\sin^{2q} a}{a^{2q}}\, da \geqslant A' > 0$$

and

$$\int_0^\pi \left\{\frac{\sin\left(\dfrac{j+1}{2}\right) a}{\sin\dfrac{a}{2}}\right\}^{2q} da \geqslant A \int_0^{1/j} \left\{\frac{ja}{a}\right\}^{2q} da \geqslant A' j^{2q-1}.$$

Thus, we get a minorant of I_j:

$$I_j \geqslant A j^{-qn} j^{(2q-1)(n-1)},$$

from which we conclude that $I_j \to \infty$ if $qn - 2q - n + 1 > 0$, i.e., $n - 1 > p \geqslant 1$.

For the second part of (2), it is sufficient to prove

$$\text{(A)} \qquad \sup_{j,x} |\sigma_{(j,\ldots,j)}(x, \phi f)| \leqslant A\|f\|_p, \qquad n-1 < p < \infty,$$

where $|x| \leqslant \epsilon'$, ϵ' being fixed with $0 < \epsilon' < \epsilon$. In fact, let f be a function of $L^p(T_n)$ such that $f = 0$ if $|x| < \delta$, and let g be a function of $C^\infty(T_n)$ satisfying $\|f - g\|_p < \eta$ and $g = 0$ if $|x| < \epsilon$. Then $\lim_{j\to\infty} \sigma_{(j,\ldots,j)}(x, g) = 0$, uniformly in $|x| \leqslant \epsilon'$, and

$$\sup_j |\sigma_{(j,\ldots,j)}(x, f-g)| = \sup_j |\sigma_{(j,\ldots,j)}(x, \phi(f-g))| \leqslant A\|f-g\|_p < A\eta$$

for $|x| \leqslant \epsilon'$. Thus, $\lim_{j\to\infty} \sigma_{(j,\ldots,j)}(x, f) = 0$, uniformly in $|x| \leqslant \epsilon'$.

Now we prove (A). By Hölder's inequality, $|\sigma_{(j,\ldots,j)}(x, \phi f)| \leqslant \|f\|_p J_j^{1/q}$, where

$$J_j = \int_{T_n} |\phi(y) K_{(j,\ldots,j)}(x-y)|^q \, dy.$$

Thus

$$J_j \leqslant \frac{A}{j^{nq}} \int_\alpha^\pi \left\{\frac{\sin \frac{1}{2}(j+1) y_1}{\sin \frac{1}{2} y_1}\right\}^{2q} dy_1 \int_0^\pi \left\{\frac{\sin \frac{1}{2}(j+1) y_2}{\sin \frac{1}{2} y_2}\right\}^{2q} dy_2 \cdots \int_0^\pi \left\{\frac{\sin \frac{1}{2}(j+1) y_n}{\sin \frac{1}{2} y_n}\right\}^{2q} dy_n$$

where $\alpha = (\epsilon - \epsilon')/\sqrt{n}$. The first integral is finite and the others are dominated by

$$\int_0^{1/j} \frac{(ja)^{2q}}{a^{2q}} da + \int_{1/j}^\pi \frac{1}{a^{2q}} da \leqslant A j^{2q-1}$$

except for a constant multiplier. Thus we get $J_j \leqslant A j^{-qn+(2q-1)(n-1)}$, which is bounded if $p \geqslant n - 1$. Thus, (2) is proved.

3. Now we try to consider the (C, α)-sum case. The (C, α)-mean of the Dirichlet kernel is by definition

$$K_j^\alpha(t) = \sum_{\nu=0}^{j} A_{j-\nu}^{\alpha-1} D_\nu(t) / A_j^\alpha,$$

where j, t are scalars and $A_j^\alpha = \binom{j+\alpha}{j}$. We have $|K_j^\alpha(t)| \leqslant j+1$. We first assume that $0 < \alpha < 1$. By an elementary calculation we get

$$K_j^\alpha(t) = \frac{1}{2A_j^\alpha \sin\frac{t}{2}} \mathscr{T}_m \left\{ \sum_{\nu=0}^{j} A_{j-\nu}^{\alpha-1} \exp\left[i(\nu+\tfrac{1}{2})t\right] \right\}$$

$$= \mathscr{T}_m \left\{ \frac{\exp\left[i(j+\frac{1}{2})t\right]}{2A_j^\alpha \sin\frac{t}{2}} \sum_{\nu=0}^{j} A_\nu^{\alpha-1} e^{-i\nu t} \right\}$$

$$= \mathscr{T}_m \left\{ \frac{\exp\left[i(j+\frac{1}{2})t\right]}{2A_j^\alpha \sin\frac{t}{2}} \left[\frac{1}{(1-e^{-it})^\alpha} - \sum_{\nu=j+1}^{\infty} A_\nu^{\alpha-1} e^{-i\nu t} \right] \right\}$$

$$= \frac{\sin\left(j+\frac{1}{2}+\frac{\alpha}{2}\right)t}{A_j^\alpha \left(2\sin\frac{t}{2}\right)^{\alpha+1}} + H_j^\alpha(t), \quad \text{say.}$$

Since $A_\nu^{\alpha-1}$ decreases monotonically to zero, the last sum converges in $0 < |t| < \pi$, and by summation by parts, it does not exceed in absolute value $2A_{j+1}^{\alpha-1}|1-e^{-it}|^{-1}$. Thus,

$$|H_j^\alpha(t)| \leqslant \frac{A_{j+1}^{\alpha-1}}{A_1^\alpha} \frac{1}{(\sin\frac{1}{2}t)^2} \leqslant \frac{A}{jt^2} \leqslant AC^{\alpha-1} j^{-\alpha} t^{-\alpha-1},$$

if $jt > C$. Since $0 < \alpha < 1$, $C^{\alpha-1}$ is small for large C. Therefore, if $I_j(\alpha)$ are defined for (C, α)-kernels analogously to I_j, we get in a similar way,

$$I_j(\alpha) \geqslant Aj^{(n-1)(q-1)-\alpha q}.$$

In fact, we have

$$\left\{\int_\delta^\pi |K_j^\alpha(t)|^q\, dt\right\}^{1/q} \geqslant \frac{A}{j^\alpha}\left\{\int_\delta^\pi \frac{\sin^q\left(j+\frac{1}{2}+\frac{\alpha}{2}\right)t}{t^{(\alpha+1)q}}\, dt\right\}^{1/q} - \frac{A'}{j^\alpha}\left\{\int_\delta^\pi \frac{dt}{t^{(\alpha+1)q}}\right\}^{1/q}$$

$$\geqslant A'' j^{-\alpha},$$

since we can take A' sufficiently small for large j. We have

$$\int_0^\pi |K_j^\alpha(t)|^q\, dt \geqslant \int_0^{1/j} j^q\, dt = Aj^{q-1}.$$

In the same way as before, we get $J_j(\alpha) \leqslant Aj^{(n-1)(q-1)-\alpha q}$ for the (C, α)-analogue of J_j. Therefore we conclude

THEOREM 2. *Let* $0 < \alpha < 1$. *Then square* (C, α)*-sums have L.P. for* $L^p(T_n)$ *if* $p \geqslant (n-1)/\alpha$, *but not if* $(n-1)/\alpha > p \geqslant 1$.

Remark 2. As is easily seen, in (2) of Theorem 1, we can replace (C, 1)-sums by Abel means. From this fact and from (2) of Theorem 1, if $1 \leqslant \alpha < \infty$, square (C, α)-means have L.P. for $L^p(T_n)$ if $p \geqslant n-1$, but do not have it if $n-1 > p \geqslant 1$.

Remark 3. The case $p = 1$ is known, see [*1*] and [*2*].

4. As a consequence of Theorem 1, we state an analogue of Lebesgue's theorem.

For $f \in L^p(T_n)$, put

$$\phi_x(y) = \sum [f(x_1 \pm y_1, \ldots, x_n \pm y_n) - f(x_1, \ldots, x_n)]$$

where $\sum$ sums all possible combinations of signs, and denote

$$\Phi_{x,p}(t) = \Phi(t) = \left(\int_{|y| \leqslant t} |\phi_x(y)|^p \, dy\right)^{1/p}.$$

THEOREM 3. *If* $f \in L^p(T_n)$, $p > n-1$, *then its Fourier series is square* (C, 1)*-summable to* $f(x)$ *at* x *where* $\Phi(t) = o(t^{n/p})$ $(t \to 0)$.

Proof. We first note that

$$\sigma_{(j,\ldots,j)}(x,f) - f(x) = \frac{1}{\pi^n} \int_0^\pi \cdots \int_0^\pi \phi_x(y) \, K_{(j,\ldots,j)}(y) \, dy.$$

Put $E_\nu = \{y = (y_1, \ldots, y_n) \colon y_1, \ldots, y_n \geqslant 0, 2^{-\nu} \leqslant |y| < 2^{-\nu+1}\}$. Then by Theorem 1, the repeated integral equals

$$\left\{\sum_{\nu=N}^{M} \int_{E_\nu} + \int_{|y| \leqslant 2^{-M}}\right\} \phi_x(y) \, K_{(j,\ldots,j)}(y) \, dy + o(1),$$

where N is sufficiently large but fixed, and M is chosen so that $2^M \leqslant j < 2^{M+1}$ Since $K_{(j,\ldots,j)}(y) \leqslant (j+1)^n$, the second integral is $O(1)\Phi(1/j)\, j^{n/p} = o(1)$. If $y \in E_\nu$, then at least one of the y_i satisfies $2^{-\nu}/\sqrt{n} \leqslant y_i \leqslant 2^{-\nu+1}$. Thus the integral over E_ν is dominated by

$$O(1) \frac{2^{2\nu}}{j^n} \sum_{k=1}^{n} \int_{E_\nu} |\phi_x(y)| \prod_{i \neq k} \left\{\frac{\sin (j + 1/2) y_i}{\sin y_i/2}\right\}^2 dy$$

$$\leqslant O(1) \frac{2^{2\nu}}{j^n} \Phi(2^{-})^{\nu} \, 2^{-\nu/q} \prod_{i=1}^{n-1} \left[\int_0^\pi \left\{\frac{\sin (j + 1/2) \, y_i}{\sin y_i/2}\right\}^{2q} dy\right]^{1/q}$$

$$= o(1) \, j^{\{(n-1)/p-1\}} \, 2^{-\nu\{(n-1)/p-1\}}.$$

Thus, summing over ν, we get, finally, $\sigma_{(j,\ldots,j)}(x,f) - f(x) = o(1)$ as $j \to \infty$.

Remark 4. If $f \in L^p(T_n)$, $p \geqslant 1$, then $\Phi_{x,p}(t) = o(t^{n/p})$ almost everywhere. But it is known that $\sigma_{(j,\ldots,j)}(x,f)$ tends to $f(x)$ almost everywhere if $f \in L^1(T_n)$ (see [3]).

References

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